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Capillary instability of a rotating jet with twisted magnetic field

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The stability of a uniformly rotating and infinitely conducting jet with twisted magnetic field is discussed. The dispersion relation is obtained for the non-axisymmetric perturbations; this involves a complex and implicit function of the growth rate. The complex growth rate implies that instability develops as an overstability. The effect of axial current is destabilizing for both the instabilities $m = 0$ and $m = 1$ discussed here. The effect of rotation on the stability of the jet is as follows: (*a*) for the $m = 0$ mode, rotation always has a destabilizing effect; (*b*) for the $m = 1$ mode, rotation has a stabilizing effect for very long-wave perturbations, but for short-wave perturbations it destabilizes the jet. In particular there exists a disturbance of a particular wavelength which is unaffected by the rotation of the jet.

1. Introduction

Rayleigh (1899) has discussed the capillary instability of a cylindrical jet which is at rest and subjected to axisymmetric disturbances. He showed that the jet is unstable to perturbations having wavelength greater than the circumference of the cross-section. He also developed for the first time the important concept of mode of maximum instability. Later Dattner, Lehnert & Lundquist (1958) performed an experiment with a column of liquid mercury, carrying an axial volume current, and observed an instability for the mode $m = 0$. Murty (1960) then presented a theoretical study of the same problem for very small values of electric conductivity and estimated the growth rate of instability. Later Tayler (1960) showed that for finite conductivity the axial current has a destabilizing effect for all wavelengths for a simple jet. Chandrasekhar (1961) then discussed the problem of a finitely conducting liquid jet carrying longitudinal field for axisymmetric perturbations, and the non-axisymmetric perturbations were discussed by Nayyar & Trehan (1963). Recently Gupta (1964) has discussed the capillary instability of a liquid column of a conducting fluid carrying an axial volume current for the axisymmetric perturbations. It turns out that a mode of maximum instability always exists for any value of the conductivity. If in addition a longitudinal magnetic field is present, it is shown that it is possible to stabilize 'varicose' deformations of all wavelengths if the magnetic field exceeds a critical value depending on the current strength. The effect of rotation on the capillary instability of a jet does not appear to have been studied and hence in this paper we study the effect of rotation on the jet carrying volume current and placed in a longitudinal magnetic field. We have also taken into consideration the surface currents present due to discontinuity of the fields inside and outside the jet.

2. The equilibrium configuration

Let us consider an infinitely long, inviscid incompressible and perfectly conducting liquid jet rotating with an angular velocity $\mathbf{\Omega}$ about its axis and surrounded by vacuum. The twisted magnetic field configuration is $(0, Ar/R, B)$ inside and $(0, A_0R/r, B_0)$ outside the jet, R being the radius of the jet. In equilibrium, the pressure inside the rotating jet is given by

$$p = \frac{T}{R} + \left[\frac{A^2}{4\pi R^2} - \rho\frac{\Omega^2}{2}\right][R^2 - r^2] + \frac{1}{8\pi}[A_0^2 + B_0^2 - A^2 - B^2], \tag{1}$$

where T denotes the surface tension.

3. Normal—mode analysis

(a) Perturbed equations and their solutions inside the jet

Let the cylinder be perturbed by a cylindrical wave perturbation of the form

$$\boldsymbol{\xi}(r)\exp\{im\theta + ikz + \sigma t\}$$

of amplitude $\boldsymbol{\xi}(r)$, azimuthal wave-number m, longitudinal wave-number k and time constant σ^{-1}, where m, k are real. The linearized perturbed equations of the problem become

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\operatorname{grad}\delta p + \frac{\delta\mathbf{j}}{c}\times\mathbf{B} + \frac{\mathbf{j}}{c}\times\delta\mathbf{B} + 2\rho\,(\mathbf{v}\times\mathbf{\Omega}), \tag{2}$$

$$\operatorname{div}\boldsymbol{\xi} = 0, \tag{3}$$

$$\operatorname{div}\delta\mathbf{B} = 0, \tag{4}$$

$$\delta\mathbf{B} = \operatorname{curl}(\boldsymbol{\xi}\times\mathbf{B}). \tag{5}$$

After simplification and reduction, the solutions inside the jet may be expressed in the form

$$\xi_r = \frac{-ik}{K}C_1\left[\beta\frac{m}{x}I_m(x) + I_m'(x)\right], \tag{6_r}$$

$$\xi_\theta = \frac{k}{K}C_1\left[\frac{m}{x}I_m(x) + \beta I_m'(x)\right], \tag{6_θ}$$

$$\xi_z = C_1 I_m(x), \tag{6_z}$$

$$\delta\mathbf{B} = iC\boldsymbol{\xi}, \tag{7}$$

$$\left(\frac{\delta p}{\rho}\right)^i = \frac{-i}{4\pi\rho k}\left[\left(\frac{CAm}{R} + 4\pi\rho\sigma^2\right)(-\xi_z) + \frac{CAkr}{R}\xi_\theta\right], \tag{8}$$

where
$$x = Kr, \quad K^2 = (1-\beta^2)\,k^2,$$
$$C = kB + \frac{Am}{R},$$
$$\beta = [2(CA/R) + 4\pi\rho(2i\sigma\Omega)]/(C^2 + 4\pi\rho\sigma^2), \tag{9}$$
and C_1 is an arbitrary constant.

(*b*) *Perturbed equations and their solutions outside the jet*

The solutions for the exterior vacuum are
$$\delta\mathbf{B}_0 = C_2\left[kK'_m(y), \frac{im}{r}K_m(y), ik\,K_m(y)\right], \tag{10}$$
where $y = kr$ for $r > R$, and C_2 is an arbitrary constant.

3. Boundary conditions connecting the inside and outside solutions on the perturbed surface of the jet

First, the normal component of the magnetic field must be continuous, i.e.
$$\mathbf{n}.[\delta\mathbf{B}] + \delta\mathbf{n}.[\mathbf{B}] = 0,$$
where the motion of the unit normal $\mathbf{n}$ is determined in terms of the perturbed velocity $\mathbf{v}$ by the equation
$$\frac{\partial\mathbf{n}}{\partial t} + \mathbf{v}.\nabla\mathbf{n} = \mathbf{n}\times[\mathbf{n}\times\{(\nabla\mathbf{v}).\mathbf{n}\}],$$
where the normal is directed into the jet and $[X]$ denotes the jump in the physical quantity X as one passes from inside the jet to outside.

Secondly, the normal component of the stress tensor must be continuous, i.e.
$$\mathbf{j}^*\times\overline{\delta\mathbf{B}} + \delta\mathbf{j}^*\times\overline{\mathbf{B}} - \mathbf{n}[\delta p] - \delta\mathbf{n}[P] = 0,$$
where $\mathbf{j}^*$ and $\delta\mathbf{j}^*$ are surface currents on the jet and $\overline{X}$ denotes the average value of the quantity X as one passes just from inside to outside of the jet.

Using the above boundary conditions of the problem, the dispersion relation is given by
$$P_m(Y)\left[\beta m + X\frac{I'_m(X)}{I_m(X)}\right] = [1-\beta^2]\left[\left(\frac{B}{B_s}\,Y + \frac{A}{B_s}m\right)^2 + \sigma_*^2\right], \tag{11}$$
where
$$Y = kR, \quad X = KR, \quad B_s^2 = \frac{4\pi T}{R},$$
$$\sigma_*^2 = \frac{\sigma^2}{(T/\rho R^3)}, \quad \Omega_*^2 = \frac{\Omega^2}{(T/\rho R^3)},$$
and
$$P_m(Y) = (1-m^2-Y^2) + \Omega_*^2 + \frac{A_0^2 - A^2}{B_s^2} + \left(\frac{B_0}{B_s}\,Y + \frac{A_0}{B_s}m\right)^2\frac{K_m(Y)}{YK'_m(Y)}. \tag{12}$$

In the absence of rotation, and for axisymmetric perturbation the dispersion equation (11) reduces to equation (78) of Gupta (1964).

4. Discussion

The dispersion relation gives the frequency of oscillation implicitly as a complex function of the wave-number and parameters describing the basic flow. As the most general case is too complicated to be simply discussed we shall consider only two simple cases, the axisymmetric mode and the $m = 1$ mode.

Case I, $m = 0$

The dispersion relation gives

$$P_0(Y)\left[X\frac{I_1(X)}{I_0(X)}\right] = [1-\beta^2]\left[Y^2\frac{B^2}{B_s^2}+\sigma_*^2\right], \tag{13}$$

where
$$\beta = \frac{2A}{B_s}\left(Y\frac{B}{B_s}\right)+2i\sigma_*\,\Omega_*\Big/\left(Y^2\frac{B^2}{B_s^2}+\sigma_*^2\right),$$

and
$$P_0(Y) = (1-Y^2)+\Omega_*^2+\frac{A_0^2-A^2}{B_s^2}+\frac{B_0^2}{B_s^2}Y\frac{K_0(Y)}{K_0'(Y)}.$$

For moderate value of the field strength, and in the absence of any surface currents, we write our dispersion equation as

$$\left[(1-Y^2)+\Omega_*^2-\frac{B_0^2}{B_s^2}Y\frac{K_0(Y)}{K_1(Y)}\right]\left[X\frac{I_1(X)}{I_0(X)}\right] = [1-\beta^2]\left[Y^2\frac{B^2}{B_s^2}+\sigma_*^2\right]. \tag{14}$$

To discuss the effect of rotation we shall consider the dispersion equation near the origin, i.e. for $Y \to 0$ (for $Y = 0$, β becomes indeterminate and the dispersion relation does not remain valid). For $Y \to 0$ the dispersion equation reduces to

$$[1+\Omega_*^2](+\epsilon) = (1-\beta^2)\,\sigma_*^2, \tag{15}$$

where ϵ is a very small quantity, which tends to zero as $Y \to 0$. For $\beta \neq 1$, equation (15) shows that, as $\epsilon \to 0$, $\sigma_*^2 \to 0$, which implies that

$$\sigma_*^2 \to 0 \quad \text{as} \quad Y \to 0. \tag{16}$$

In the non-magnetic case at $Y = 1$ the dispersion equation gives

$$[\Omega_*^2]\left[X\frac{I_1(X)}{I_0(X)}\right] = [1-\beta^2]\,[\sigma_*^2], \tag{17}$$

where
$$\beta^2 = -4\Omega^2/\sigma^2. \tag{18}$$

Evidently equation (17) can be satisfied only for positive values of σ_*^2, since the left-hand side is a positive quantity. This result, when compared with a jet with no rotation, implies a destabilizing effect of the rotation for all values of the wave-number.

To discuss the effect of the axial current inside the jet we for simplicity make $\Omega = 0$, and dispersion equation becomes

$$\left[(1-Y^2)+\frac{A_0^2-A^2}{B_s^2}+\frac{B_0^2}{B_s^2}Y\frac{K_0(Y)}{K_0'(Y)}\right]\left[-Z\frac{J_1(Z)}{J_0(Z)}\right] = -[\beta^2-1]\left[Y^2\frac{B^2}{B_s^2}+\sigma_*^2\right] \tag{19}$$

for
$$\beta > 1, \quad Z^2 = (\beta^2-1)\,Y^2.$$

This dispersion relation has been solved numerically, for the mode $m = 0$. Curve 1 in figure 1 gives the values of σ_*^2 for $A^2 = A_0^2 = B^2 = B_0^2 = 0{\cdot}75B_s^2$, for different values of Y, and curve 2 gives the values of σ_*^2 when we assume $A = 0$. Comparison of both these curves gives us a destabilizing influence of the axial current considered here.

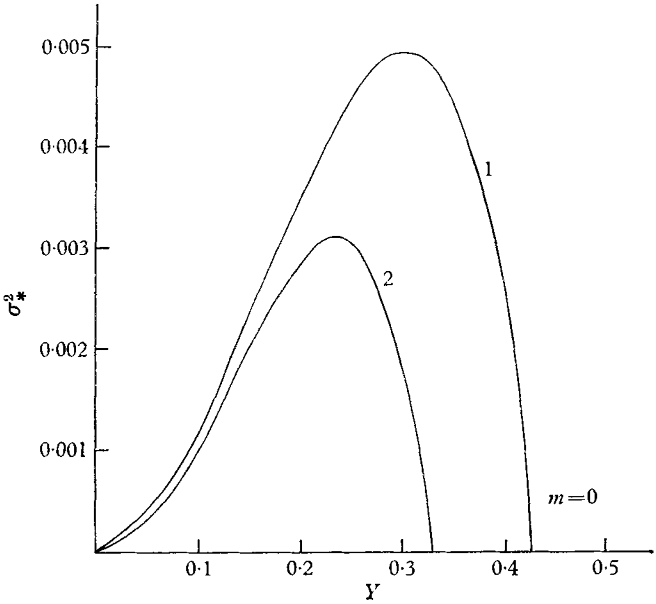

FIGURE 1. Curve 1: the dispersion relation with volume current

$(A^2 = A_0^2 = B^2 = B_0^2 = 0{\cdot}75B_s^2)$;

curve 2: the dispersion relation with sheet current

$(A = 0,\quad A_0^2 = B^2 = B_0^2 = 0{\cdot}75B_s^2)$.

The abscissa measures the wave-number in the unit R^{-1} and the ordinate the square of growth rate in the unit $(T/\rho R^3)$.

Again if we consider both rotation and twisted magnetic field together we get β to be complex and hence X is also complex which gives rise to a complex value of the growth rate. Thus we conclude that instability caused by the uniform rotation of the jet with a twisted magnetic field will occur as an overstability.

Case II, $m = 1$

The dispersion equation in this case becomes

$$P_1(Y)\left[\beta + X\frac{I_1'(X)}{I_1(X)}\right] = [1-\beta^2]\left[\left(Y\frac{B}{B_s}+\frac{A}{B_s}\right)^2 + \sigma_*^2\right], \tag{20}$$

where
$$P_1(Y) = -Y^2 + \Omega_*^2 + \frac{A_0^2 - A^2}{B_s^2} + \left(\frac{B_0}{B_s}Y + \frac{A_0}{B_s}\right)^2 \frac{K_1(Y)}{YK_1'(Y)},$$

and
$$\beta = \left[2\frac{A}{B_s}\left(Y\frac{B}{B_s}+\frac{A}{B_s}\right) + 2i\sigma_*\Omega_*\right]\Big/\left[\left(Y\frac{B}{B_s}+\frac{A}{B_s}\right)^2 + \sigma_*^2\right].$$

For $Y = 0$ and with no surface currents we obtain $\sigma^2 = -\Omega^2$ to be a solution of the dispersion equation (20) for any value of the magnetic field. Thus the system is always stable for very long wavelengths.

For the simple case with no magnetic field we get

$$[-Y^2+\Omega_*^2]\left[\beta+X\frac{I_0(X)}{I_1(X)}-1\right] = [1-\beta^2]\,\sigma_*^2. \tag{21}$$

For $\beta \neq 1$, the left-hand side is zero only if $Y^2 = \Omega_*^2$. Thus

$$\sigma_*^2 = 0 \quad \text{for} \quad Y = \Omega_*.$$

For instance, if $\Omega_*^2 = 0{\cdot}5$, then the critical value of Y is $0{\cdot}7071$. Moreover, numerical computation has indicated that, for

$$A^2 = A_0^2 = B^2 = B_0^2 = 0{\cdot}75B_s^2 \quad \text{and} \quad \Omega_*^2 = 0{\cdot}5,$$

the growth rate σ_*^2 is zero at $Y = 0{\cdot}29$. Thus we conclude that the magnetic field configuration considered here has a stabilizing effect on the jet, though the effect of volume current only is destabilizing.

The effect of rotation is: (i) for very large values of the wavelengths, rotation has a stabilizing effect on the jet; (ii) for short wavelengths it is destabilizing. Thus there always exists a disturbance of a particular wavelength, λ_{rot} say, which is unaffected by the rotation; λ_{rot} is a function of the parameters of the system considered.

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